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ON PSEUDO-FOURIER-MEHLER TRANSFORMS AND INFINITESIMAL GENERATORS IN WHITE NOISE CALCULUS*)

ISAMU DÔKU (道工 勇)

Department of Mathematics
Saitama University
Urawa 338 Japan
e-mail: doku@sacs.sv.saitama-u.ac.jp

§1. Introduction

The study of the Fourier transform $\mathcal{F}$ in white noise calculus was initiated and has been developed to a mature level by H.-H. Kuo [16,17] (also [19]). While, the Fourier-Mehler transform $\mathcal{F}_\theta$ is a kind of generalization of $\mathcal{F}$ [18] (also [11]), which furnishes the theory of infinite dimensional Fourier transforms in white noise space with adequately fruitful and profitable ingradients.

In this article we introduce Pseudo-Fourier-Mehler (PFM for short) transform having quite similar nice properties as the Fourier-Mehler transform possesses. It was originally defined in [5] and used for application to abstract equations in infinite dimensional spaces. In connection with other Fourier type transforms in white noise analysis, we can compute the infinitesimal generator of the PFM transform directly and show that our Pseudo-Fourier-Mehler transform enjoys intertwining properties. We shall state the characterization theorem for PFM transforms, which is one of our main results in this article. The Fock expansion of PFM transform can be derived as well. Lastly we shall introduce a generalization idea of PFM transform and investigate some properties that the generalized transform should satisfy.

The Pseudo-Fourier-Mehler transform is a very important and interesting operator in the standpoint of how to express the solutions for the Fourier-transformed abstract Cauchy problems ([5,6]; see also [4,8]).

In [1] they have studied the two dimensional complex Lie group $\mathcal{G}$ explicitly and succeeded in describing every one parameter subgroup with infinitesimal generator $(\frac{2a+b}{2})\Delta_G + bN$, where N is the number operator and Δ_G is the Gross Laplacian. Furthermore, one can find in [24] another related work, especially on a systematic study of Lie algebras containing infinite dimensional Laplacians.

We are able to state our results in the general setting (e.g., [23]; see also [7]) of white noise analysis. As a matter of fact, almost all statements in our theory

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remains valid under non-minor change of the basic setting. However, just for simplicity we adopt in this article the so-called original standard setting [11] in white noise analysis or Hida calculus to state our results related to the PFM transform.

§2. Notation and Preliminaries

Let $\mathcal{S} \equiv \mathcal{S}(\mathbb{R})$ be the Schwartz class space on $\mathbb{R}$ and $\mathcal{S}^* \equiv \mathcal{S}'(\mathbb{R})$ its dual space. Then $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ is a Gelfand triple. We define the family of norms given by $|\xi|_p = |A^p\xi|$, $p > 0$, $\xi \in \mathcal{S}(\mathbb{R})$, where the operator $A = -d^2/dt^2 + t^2 + 1$ and $|\cdot|$ is the $L^2(\mathbb{R})$-norm. Let $\mathcal{S}_p \equiv \mathcal{S}_p(\mathbb{R})$ be the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm $|\cdot|_p, p > 0$. We denote its dual space by $\mathcal{S}_p^* \equiv \mathcal{S}_p'(\mathbb{R})$, and we have $\mathcal{S}_p(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}_p'(\mathbb{R})$. Let μ be the standard Gaussian measure on $\mathcal{S}'(\mathbb{R})$ such that

$$\int_{\mathcal{S}^*} \exp(\sqrt{-1}\langle x, \xi\rangle)\mu(dx) = \exp\Big(-\frac{1}{2}|\xi|^2\Big),$$

for any $\xi \in \mathcal{S}(\mathbb{R})$. (L^2) denotes the Hilbert space of complex-valued μ-square integrable functionals with norm $\|\cdot\|$. The Wiener-Itô decomposition theorem gives the unique representation of φ in (L^2), i.e.,

$$\varphi = \sum_{n=0}^{\infty} I_n(f_n), \qquad f_n \in \hat{L}^2_{\mathbb{C}}(\mathbb{R}^n), \tag{1}$$

where I_n denotes the multiple Wiener integral of order n and $\hat{L}^2_{\mathbb{C}}(\mathbb{R}^n)$ the space of symmetric complex valued L^2-functions on $\mathbb{R}^n$. The second quantization operator $\Gamma(A)$ is densely defined on (L^2) as follows: for $\varphi = \sum_{n=0}^{\infty} I_n(f_n) \in \mathrm{Dom}(\Gamma(A))$,

$$\Gamma(A)\varphi = \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n). \tag{2}$$

For $p \in \mathbb{N}$, define $\|\varphi\|_p = \|\Gamma(A)^p\varphi\|$ and let $(\mathcal{S})_p \equiv \{\varphi \in (L^2); \|\varphi\|_p < \infty\}$ and the dual space of $(\mathcal{S})_p$ is denoted by $(\mathcal{S})_p^*$. Let $(\mathcal{S})$ be the projective limit of $\{(\mathcal{S})_p; p \in \mathbb{N}\}$. It is called a space of test white noise functionals. The elements in the dual space $(\mathcal{S})^*$ of $(\mathcal{S})$ are called generalized white noise functionals or Hida distributions. In fact, $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$ is a Gelfand triple [11]. For convention all dual pairings $\langle\cdot,\cdot\rangle$, resp. $\langle\langle\cdot,\cdot\rangle\rangle$ mean the canonical bilinear forms on $\mathcal{S}^* \times \mathcal{S}$ (resp. $(\mathcal{S})^* \times (\mathcal{S})$) unless otherwise stated.

The S-transform of $\Phi \in (\mathcal{S})^*$ is a function on $\mathcal{S}$ defined by

$$(S\Phi)(\xi) := \langle\langle \Phi, :\exp\langle\cdot,\xi\rangle:\rangle\rangle, \qquad \xi \in \mathcal{S}(\mathbb{R}), \tag{3}$$

where $:\exp\langle\cdot,\xi\rangle: \equiv \exp\langle\cdot,\xi\rangle \cdot \exp\left(-\frac{1}{2}|\xi|^2\right)$. Then note that a mapping : $\mathbb{C} \ni z \mapsto (S\Phi)(z\xi+\eta)$ is entire holomorphic for any $\xi, \eta \in \mathcal{S}$. A complex valued function F on $\mathcal{S}$ is called a *U-functional* if and only if it is ray entire on $\mathcal{S}$ and if there exist constants $C_1, C_2 > 0$, and $p \in \mathbb{N} \cup \{0\}$ so that the estimate

$$|F(z\xi)| \le C_1 \exp\Big(C_2|z|^2|\xi|_p{}^2\Big)$$

may hold for all $z \in \mathbb{C}, \xi \in \mathcal{S}$. We have the following Characterization Theorem [25]:

Theorem 1. *If $\Phi \in (\mathcal{S})^*$, then $S\Phi$ is a U-functional. Conversely, if F is a U-functional, then there exists a unique element Φ in $(\mathcal{S})^*$ such that $S\Phi = F$ holds.*

Based upon the above characterization we are able to give rigorous definitions to Fourier type transforms of infinite dimensions. The Kuo type Fourier transform $\mathcal{F}$ [16,17] of a generalized white noise functional Φ in $(\mathcal{S})^*$ is the generalized white noise functional, S-transformation of which is given by

$$S(\mathcal{F}\Phi)(\xi) = \langle\langle \Phi, \exp(-i\langle \cdot, \xi\rangle)\rangle\rangle, \qquad \xi \in \mathcal{S}. \tag{4}$$

Likewise, the Fourier-Mehler transform $\mathcal{F}_\theta$ ($\theta \in \mathbb{R}$) [18] of a generalized white noise functional Φ in $(\mathcal{S})^*$ is the generalized white noise functional, S-transformation of which is given by

$$S(\mathcal{F}_\theta\Phi)(\xi) = \langle\langle \Phi, \exp\Big\{ \mathrm{e}^{i\theta}\langle \cdot, \xi\rangle - \frac{1}{2}\mathrm{e}^{i\theta}\cos\theta|\xi|^2\Big\}\rangle\rangle, \qquad \xi \in \mathcal{S}. \tag{5}$$

The Fourier-Mehler transform $\mathcal{F}_\theta, \theta \in \mathbb{R}$ is a generalization of the Kuo type Fourier transform $\mathcal{F}$. Actually, $\mathcal{F}_0 = Id$, and $\mathcal{F}_{-\pi/2}$ is coincident with the Fourier transform $\mathcal{F}$. It is easy to see that $\mathcal{F}_{\pi/2}$ is the inverse Fourier transform $\mathcal{F}^{-1}$. Hence we have

$$S(\mathcal{F}^{-1}\Phi)(\xi) = (S\Phi)(i\xi)\exp\Big(-\frac{1}{2}|\xi|^2\Big), \qquad \xi \in \mathcal{S}.$$

§3. Pseudo-Fourier-Mehler Transform

We begin with introducing the Pseudo-Fourier-Mehler transform in white noise analysis.

Definition 1. *$\{\Psi_\theta, \theta \in \mathbb{R}\}$ is said to be the Pseudo-Fourier-Mehler (PFM) transform [5,6] if Ψ_θ is a mapping from $(\mathcal{S})^*$ into itself for $\theta \in \mathbb{R}$, whose U-functional is given by*

$$S(\Psi_\theta\Phi)(\xi) = F(e^{i\theta}\xi)\cdot\exp\Big(ie^{i\theta}\sin\theta|\xi|^2\Big), \qquad \xi \in \mathcal{S}, \tag{6}$$

or equivalently

$$S(\Psi_\theta\Phi)(\xi) = \langle\langle \Phi, \exp\Big(e^{i\theta}\langle \cdot, \xi\rangle - \frac{1}{2}|\xi|^2\Big)\rangle\rangle, \qquad \xi \in \mathcal{S}, \tag{7}$$

for $\Phi \in (\mathcal{S})^*$, where S is the S-transform in white noise analysis and F denotes the U-functional of Φ.

By virtue of Theorem 1, the right hand sides in Eq.(6) and Eq.(7) are U-functionals, and $\Psi_\theta\Phi$ exists for each Φ in $(\mathcal{S})^*$. Therefore the above-mentioned Pseudo-Fourier-Mehler transform is well-defined. Hence we have

Proposition 2. *The following properties hold:*
(i) $\Psi_0 = Id$; *(Id denotes the identity operator.)*
(ii) $\Psi_\theta \neq \mathcal{F}$ *for any* $\theta \in \mathbb{R} \setminus \{0\}$;
(iii) $\Psi_\theta \neq \mathcal{F}_\theta$ *for any* $\theta \in \mathbb{R} \setminus \{0\}$.

Proof. As to (i), it is easy to see that $S(\Psi_0\Phi)(\xi) = S\Phi(\xi) = F(\xi)$. The characterization theorem allows the equality $\Psi_0 = Id$. (iii) is obvious from definitions. Since $\mathcal{F}_0 = Id$ and $\mathcal{F}_{-\pi/2} = \mathcal{F}$, it follows clearly from (iii) that $\mathcal{F}$ never coincides with Ψ_θ for any $\theta \in \mathbb{R}$ except $\theta = 0$. □

Proposition 3. *The invese operator of the Pseudo-Fourier-Mehler transform* Ψ_θ *is given by* $(\Psi_\theta)^{-1} = \Psi_{-\theta}$ *for* $\theta \in \mathbb{R}$.

Proof. It is sufficient to show that $\Psi_{-\theta}\Psi_\theta = \Psi_\theta\Psi_{-\theta} = Id$. As a matter of fact, for $\Phi \in (\mathcal{S})^*$ we get from the definition (6)

$$
\begin{aligned}
(8) \qquad S(\Psi_{-\theta}(\Psi_\theta\Phi))(\xi) &= S(\Psi_\theta\Phi)(\mathrm{e}^{-i\theta}\xi) \cdot \exp\Big(-ie^{-i\theta}\sin\theta|\xi|^2\Big) \\
&= (S\Phi)(\mathrm{e}^{i\theta}(\mathrm{e}^{-i\theta}\xi)) \cdot \exp\Big(ie^{i\theta}\sin\theta|\mathrm{e}^{-i\theta}\xi|^2\Big) \cdot \exp\Big(-ie^{-i\theta}\sin\theta|\xi|^2\Big) \\
&= (S\Phi)(\xi) \cdot \exp(0) = S(Id \cdot \Phi)(\xi), \qquad \xi \in \mathcal{S},
\end{aligned}
$$

because we used the relation

$$
S(\Psi_{-\theta}\Phi)(\xi) = S\Phi(\mathrm{e}^{-i\theta}\xi) \cdot \exp\Big(-i\mathrm{e}^{-i\theta}\sin\theta|\xi|^2\Big)
$$

so as to obtain the second line of Eq.(8). An application of the characterization theorem to Eq.(8) gives $\Psi_{-\theta}\Psi_\theta = Id$. As for the other part of the desired equalities, it goes almost similarly. □

Next let us consider what the image of the space $(\mathcal{S})$ under Ψ_θ is like (see Corollary 6 below). The Pseudo-Fourier-Mehler transform Ψ_θ also enjoys some interesting properties on the product of Gaussian white noise functionals (see Theorem 4 and Theorem 5).

Theorem 4. *Let* g_c *be a Gaussian white noise functional, i.e.,* $g_c(\cdot) := \mathcal{N}\exp(-|\cdot|^2/2c)$ *with renormalization* $\mathcal{N}$ *and* $c \in \mathbb{C}$, $c \neq 0, -1$. *For* $\theta \in \mathbb{R}$ *the following equalities hold:*

$$
(i)\ \Psi_\theta\Phi : g_{c(\theta)} = \Gamma(e^{i\theta}Id)\Phi, \qquad \forall\Phi \in (\mathcal{S})^*;
$$

$$
(ii)\ \text{for any } p \in \mathbb{R}, \quad \|\Psi_\theta\Phi : g_{c(\theta)}\|_p = \|\Phi\|_p, \quad \forall\Phi \in (\mathcal{S})_p;
$$

where : *denotes the Wick product (e.g. [11,p.101]) and the parameter* $c(\theta)$ *is given by* $c(\theta) = -(2^{-1}i\ e^{-i\theta}\csc\theta + 1)$.

Proof. Noting that the U-functional of g_c is given by $\exp(-2^{-1}\ (1+c)^{-1}\ |\xi|^2)$, we readily obtain

$$
\begin{aligned}
(9) \qquad S(\Psi_\theta\Phi : g_{c(\theta)})(\xi) &= S(\Psi_\theta\Phi)(\xi) \cdot (Sg_{c(\theta)})(\xi) \\
&= S\Phi(\mathrm{e}^{i\theta}\xi) \cdot \Xi(\theta, \xi), \qquad \xi \in \mathcal{S},
\end{aligned}
$$

because we employed Eq.(6) and put

$$\Xi(\theta,\xi) := \exp\left(ie^{i\theta}\sin\theta|\xi|^2 - \frac{1}{2(1+c(\theta))}|\xi|^2\right).$$

Then we cannot find any $\theta \in \mathbb{R}$ such that

$$(9) = S\Phi(\xi) = \exp\left(-\frac{1}{2}|\xi|^2\right)\cdot\langle\langle\Phi, e^{\langle\cdot,\xi\rangle}\rangle\rangle$$

may hold, which implies that $\Psi_\theta\Phi : g_{c(\theta)} \neq \Phi$ for any $\Phi \in (\mathcal{S})^*$. However, when $\Phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, f_n\rangle, f_n \in \hat{\mathcal{S}}_{-p}(\mathbb{R}^n)$ (the symmetric space $\mathcal{S}_{-p}(\mathbb{R}^n)$), then its U-functional $S\Phi(\xi)$ is given by $\sum_{n=0}^{\infty}\langle\xi^{\otimes n}, f_n\rangle$, so that, we easily get from definition of the second quantization operator Γ

$$\text{r.h.s. of } (9) = \sum_{n=0}^{\infty}\langle(e^{i\theta})^n\xi^{\otimes n}, f_n\rangle\cdot\Xi(\theta,\xi) = S(\Gamma(e^{i\theta}Id)\Phi)(\xi)\cdot\Xi(\theta,\xi).$$

Hence, if $2i(1+c(\theta))e^{i\theta}\sin\theta = 1$ holds, then clearly $\Xi(\theta,\xi)$ proves to be 1, suggesting with the characterization theorem that

$$\Psi_\theta\Phi : g_{c(\theta)} = \Gamma(e^{i\theta}Id)\Phi.$$

Moreover, it is easy to see that

$$\|\Psi_\theta\Phi : g_{c(\theta)}\|_p = \|\Gamma(e^{i\theta}Id)\Phi\|_p = \|\Phi\|_p$$

holds for any $p \in \mathbb{R}$. □

If we take the assertion obtained in Theorem 4 into account, then the following questions will arise naturally: whether the PFM transformed Φ (i.e. $\Psi_\theta\Phi$) can be represented by the Wick product of something like a transformed Φ and a Gaussian white noise functional g_c; furthermore, if so, what is the parameter $c = c(\theta)$ then? First of all, on the assumption that $\Phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, f_n\rangle \in (\mathcal{S})^*$, a simple computation gives, for $\xi \in \mathcal{S}$

$$\begin{aligned}(10)\qquad S(\Psi_\theta\Phi)(\xi) &= S\Phi(e^{i\theta}\xi)\cdot\exp\left(ie^{i\theta}\sin\theta|\xi|^2\right)\\ &= S(\Gamma(e^{i\theta}Id)\Phi)(\xi)\cdot\exp\left(ie^{i\theta}\sin\theta|\xi|^2\right).\end{aligned}$$

We know from Eq.(10) that there is no possibility that $\Psi_\theta\Phi$ may coincide with $\Phi : g_{K(\theta)}$ even for any $K(\theta), \theta \in \mathbb{R}$, because

$$(11)\qquad S(\Phi : g_{K(\theta)})(\xi) = S\Phi(\xi)\cdot(Sg_{K(\theta)})(\xi) = S\Phi(\xi)\cdot\Lambda(K(\theta),\xi)$$

with

$$\Lambda(r,\xi) := \exp\left\{-\frac{1}{2(1+r)}|\xi|^2\right\}.$$

On the other hand, since the S-transform of $\Gamma(\mathrm{e}^{i\theta})\Phi : g_{K(\theta)}$ is given by

$$S(\Gamma(\mathrm{e}^{i\theta}Id)\Phi)(\xi)\cdot\Lambda(K(\theta),\xi),$$

it is true from (10) that

$$\Psi_\theta\Phi=\Gamma(\mathrm{e}^{i\theta}Id)\Phi : g_{K(\theta)}$$

may possibly hold for $\Phi\in(\mathcal{S})^*, \theta\in\mathbb{R}$ as far as

$$2i(1+K(\theta))\mathrm{e}^{i\theta}\sin\theta+1=0$$

is satisfied. Let us next consider the evaluation of the term $\Psi_\theta\Phi$ ($\Phi\in(\mathcal{S})_p$) relative to the $(\mathcal{S})_p$-norm ($p\in\mathbb{R}$). We need to determine the parameter $A(\theta)$, which comes from the relation between $\Gamma(\mathrm{e}^{i\theta})\Phi : g_{K(\theta)}$ and $\Gamma(\mathrm{e}^{i\theta})(\Phi : g_{A(\theta)})$. By a similar calculation in (10) we readily obtain

$$\begin{aligned}(12)\qquad S\big(\Gamma(\mathrm{e}^{i\theta}Id)(\Phi : g_{A(\theta)})\big)(\xi)&=S(\Phi : g_{A(\theta)})(\mathrm{e}^{i\theta}\xi)\\&=(S\Phi)(\mathrm{e}^{i\theta}\xi)\cdot\Lambda(A(\theta),\mathrm{e}^{i\theta})\\&=(S\Phi)(\mathrm{e}^{i\theta}\xi)\cdot\exp\left\{-\frac{\mathrm{e}^{2i\theta}}{2(1+A(\theta))}|\xi|^2\right\},\end{aligned}$$

by making use of Eq.(11). A comparison of (12) with $S(\Gamma(\mathrm{e}^{i\theta})\Phi)(\xi)\cdot\Lambda(K(\theta),\xi)$ provides with

$$\Gamma(\mathrm{e}^{i\theta}Id)\Phi : g_{K(\theta)}=\Gamma(\mathrm{e}^{i\theta}Id)(\Phi : g_{A(\theta)})$$

as far as $A(\theta)=2^{-1}i\mathrm{e}^{-i\theta}\csc\theta-1$. It therefore follows that

$$\begin{aligned}\|\Psi_\theta\Phi\|_p&=\|\Gamma(\mathrm{e}^{i\theta}Id)\Phi : g_{K(\theta)}\|_p\\&=\|\Gamma(\mathrm{e}^{i\theta}Id)(\Phi : g_{A(\theta)})\|_p=\|\Phi : g_{A(\theta)}\|_p\end{aligned}$$

for all $\Phi\in(\mathcal{S})_p, p\in\mathbb{R}$, and any $\theta\in\mathbb{R}$. Summing up, we thus obtain

Theorem 5. *The following equalities hold for any $\theta\in\mathbb{R}$:*
(i) if $K(\theta)=2^{-1}ie^{-i\theta}\csc\theta-1$, then

$$\Psi_\theta\Phi=\Gamma(e^{i\theta}Id)\Phi : g_{K(\theta)},\qquad \Phi\in(\mathcal{S})^*;$$

(ii) if $A(\theta)=2^{-1}ie^{-i\theta}\csc\theta-1$, then

$$\|\Psi_\theta\Phi\|_p=\|\Phi : g_{A(\theta)}\|_p,\qquad \Phi\in(\mathcal{S})_p$$

for all $p\in\mathbb{R}$.

Let us think of the image of $\varphi\in(\mathcal{S})$ under the Pseudo-Fourier-Mehler transform. It is easily checked that $g_c : g_d=1$ holds with $c+d=-2$. So we have

$$(13)\qquad g_{c(\theta)} : g_{K(\theta)}=1.$$

From (ii) of Theorem 4, immediately, $\varphi\in(\mathcal{S})$ if and only if

$$\Psi_\theta\varphi : g_{c(\theta)}\in(\mathcal{S}),$$

so that, it is equivalent to

$$\Psi_\theta\varphi : g_{c(\theta)} : g_{K(\theta)}\in(\mathcal{S}) : g_{K(\theta)},$$

where $(\mathcal{S}) : g_{K(\theta)}$ denotes the whole space of elements $\varphi : g_{K(\theta)}$ for $\varphi\in(\mathcal{S})$. Consequently, it is obvious that $\Psi_\theta\varphi\in(\mathcal{S}) : g_{K(\theta)}$, by virtue of Eq.(13). Therefore we obtain

Corollary 6. *For $\theta \in \mathbb{R}$,*

$$\operatorname{Im} \Psi_\theta(\mathcal{S}) = (\mathcal{S}) : g_{K(\theta)} \equiv \{\varphi : g_{K(\theta)};\ \varphi \in (\mathcal{S})\},$$

where $K(\theta) = 2^{-1} i e^{-i\theta} \csc\theta - 1$.

Remark 2. The results in Theorem 4 and Theorem 5 are quite similar to those of the Fourier-Mehler transform. In fact, for $p \in \mathbb{R}$, $\Phi \in (\mathcal{S})_p$,

$$\|(\mathcal{F}_\theta \Phi) : g_{c_1(\theta)}\|_p = \|\Phi\|_p \quad \text{and} \quad \|\mathcal{F}_\theta \Phi\|_p = \|\Phi : g_{c_2(\theta)}\|_p$$

hold with $c_1(\theta) = -i\cot\theta - 2$, and $c_2(\theta) = i\cot\theta - 2$ (e.g. [11,§9.H]).

Remark 3. The image of $(\mathcal{S})$ under the Fourier-Mehler transform $\mathcal{F}_\theta$ is given by $(\mathcal{S}) : g_{i\cot\theta}$, while that of $(\mathcal{S})$ under the Fourier transform $\mathcal{F}$ coincides with the space

$$(\mathcal{S}) : \tilde{\delta}_0 \equiv \{\varphi : \tilde{\delta}_0;\ \varphi \in (\mathcal{S})\},$$

where $\tilde{\delta}_0$ is the delta function at 0 and

$$\lim_{c \to 0} g_c = \tilde{\delta}_0$$

(e.g. [11, Chapter 9]).

§4. Infinitesimal Generators

First of all, for all $\theta \in \mathcal{S}$ we define

$$\varphi_\xi(x) := \sum_{n=0}^{\infty} \frac{1}{n!} \langle : x^{\otimes n} :, \xi^{\otimes n} \rangle$$

with $x \in \mathcal{S}^*, \xi \in \mathcal{S}$. We call it an exponential vector. Then $\{G_\theta, \theta \in \mathbb{R}\}$ is an operator on $(\mathcal{S})$ defined by

$$(G_\theta \varphi_\xi)(x) := \varphi_{e^{i\theta}\xi}(x) \cdot \exp\Big(ie^{i\theta} \sin\theta |\xi|^2\Big). \tag{14}$$

Let τ denote the distribution in $(\mathcal{S} \otimes \mathcal{S})^*$ given by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \qquad \xi, \eta \in \mathcal{S}.$$

Note that it can be expressed as

$$\tau = \int_{\mathbb{R}} \delta_t \otimes \delta_t dt = \sum_{j=0}^{\infty} e_j \otimes e_j \in (\mathcal{S} \otimes \mathcal{S})^*,$$

where $\{e_n\}$ denotes a complete orthonormal basis for $L^2(\mathbb{R})$. Moreover we have

$$\tau^{\otimes n} = \int_{\mathbb{R}^n} \delta_{t_1} \otimes \delta_{t_1} \otimes \cdots \otimes \delta_{t_n} \otimes \delta_{t_n} dt_1 \cdots dt_n.$$

The following is an easy exercise. The next lemma provides with a general expression for elements of general form in $(\mathcal{S})$.

Lemma 7. *When* $\varphi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n \rangle \in (\mathcal{S})$ *with* $f_n \in \hat{\mathcal{S}}(\mathbb{R}^n)$,*(the symmetric* $\mathcal{S}(\mathbb{R}^n)$*), then* $G_\theta \varphi$ *is given by*

$$(G_\theta \varphi)(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, g_n \rangle,$$

and

$$g_n \equiv g_n(\varphi) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} (i \sin\theta)^m e^{i(n+m)\theta} \tau^{\otimes m} * f_{2m+n},$$

where for the element f_{2m+n} *in* $\hat{\mathcal{S}}(\mathbb{R}^{2m+n})$ *the term* $\tau^{\otimes m} * f_{2m+n}$ *actually has the following integral expression*

$$\begin{aligned}&(\tau^{\otimes m} * f_{2m+n})(t_1, \cdots, t_n) \\ &= \int_{\mathbb{R}^m} f_{2m+n}(s_1, s_1, \cdots, s_m, s_m, t_1, \cdots, t_n) ds_1 \cdots ds_m.\end{aligned}$$

On this account, we obtain immediately

Proposition 8. *The Pseudo-Fourier-Mehler transform* $\{\Psi_\theta; \theta \in \mathbb{R}\}$ *is given by the adjoint operator of* $\{G_\theta;\ \theta \in \mathbb{R}\}$*, i.e.,*

$$\Psi_\theta = G_\theta^*$$

holds in operator equality sense for all $\theta \in \mathbb{R}$.

The next proposition gives an explicit action of the PFM transform Ψ_θ for the generalized white noise functionals of general form. It is due to a direct computation.

Proposition 9. *For* $\Phi \in (\mathcal{S})^*$ *given as* $\Phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, F_n \rangle$*, it holds that*

$$\Psi_\theta \Phi(x) = \sum_{n=0}^{\infty} \Big\langle :x^{\otimes n}:, \sum_{l+2m=n} a(l,m,\theta) \cdot F_l \hat{\otimes} \tau^{\otimes m} \Big\rangle,$$

where the constant $a(l,m,\theta)$ *is given by*

$$a(l,m,\theta) = \frac{1}{m!} e^{i(l+m)\theta} (i \sin\theta)^m.$$

Remark 4. Similar results for Fourier-Mehler transform as the above can be found in [23]. For the proof of Proposition 9, it is almost the same as those given in [23].

It follows from Proposition 3 that the Pseudo-Fourier-Mehler transform Ψ_θ is injective and surjective. Moreover, it is easy to check that Ψ_θ is a strongly continuous operator from $(\mathcal{S})^*$ into itself, when we take Lemma 7 and Proposition 8 into consideration. Thus we have the following theorem.

Theorem 10 [5]. *The Pseudo-Fourier-Mehler transform $\Psi_\theta : (\mathcal{S})^* \to (\mathcal{S})^*$ is a bijective and strongly continuous linear operator.*

Theorem 11 [5]. *The set $\{\Psi_\theta; \theta \in \mathbb{R}\}$ forms a one parameter group of strongly continuous linear operator acting on the space $(\mathcal{S})^*$ of Hida distributions.*

Proof. For $\Phi \in (\mathcal{S})^*$, $\xi \in \mathcal{S}$, and any $\theta, \eta \in \mathbb{R}$, from (7) of Definition 1 we have

$$S(\Psi_{\theta+\eta}\Phi)(\xi) = \langle\langle \Phi, \exp\Big\{ e^{i(\theta+\eta)}\langle \cdot, \xi\rangle - \frac{1}{2}|\xi|^2 \Big\}\rangle\rangle. \tag{15}$$

While, from (6)

$$\begin{aligned} S(\Psi_\theta(\Psi_\eta))(\xi) &= S(\Psi_\eta\Phi)(e^{i\theta}\xi)\cdot \exp\Big(ie^{i\theta}\sin\theta|\xi|^2\Big) \\ &= F(e^{i\eta}(e^{i\theta}\xi))\cdot \exp\Big(ie^{i\eta}\sin\eta|e^{i\theta}\xi|^2\Big)\cdot \exp\Big(ie^{i\theta}\sin\theta|\xi|^2\Big) \\ &= e^{-\frac{1}{2}\left|e^{i(\theta+\eta)}\xi\right|^2}\langle\langle \Phi, e^{\langle\cdot, e^{i(\theta+\eta)}\xi\rangle}\rangle\rangle\cdot \exp\Big\{ie^{i\theta}\left(e^{i(\theta+\eta)}\sin\eta + \sin\theta\right)|\xi|^2\Big\} \\ &= \langle\langle \Phi, \exp\Big\{e^{i(\theta+\eta)}\langle\cdot,\xi\rangle - \frac{1}{2}|\xi|^2\Big\}\rangle\rangle, \end{aligned} \tag{16}$$

with the U-functional F of Φ. By comparing (15) with (16), we get

$$S(\Psi_{\theta+\eta}\Phi)(\xi) = S(\Psi_\theta\Psi_\eta\Phi)(\xi).$$

Consequently, the characterization theorem leads to

$$\Psi_{\theta+\eta}\Phi = \Psi_\theta\cdot\Psi_\eta\Phi, \qquad \Phi \in (\mathcal{S})^*,$$

which completes the proof. □

We are now in a position to state one of the principal results in this paper. This is a very important property of the Pseudo-Fourier-Mehler transform, especially on an applicational basis.

Theorem 12 [5]. *The infinitesimal generator of $\{\Psi_\theta; \theta \in \mathbb{R}\}$ is given by $i(N+\Delta_G^*)$, where N is the number operator and Δ_G^* is the adjoint of the Gross Laplacian Δ_G.*

Remark 5. It is well known that the infinitesimal generator of the Fourier-Mehler transforms $\{\mathcal{F}_\theta; \theta \in \mathbb{R}\}$ is $iN + \frac{i}{2}\Delta_G^*$, while the adjoint operator of $\{\mathcal{F}_\theta;\ \theta \in \mathbb{R}\}$ has $iN + \frac{i}{2}\Delta_G$ as its infinitesimal generator (e.g. see [11]). The proof of Theorem 12 is almost similsr to the above ones.

Proof of Theorem 12. First of all we set

$$F_\theta(\xi) := S(\Psi_\theta\Phi)(\xi) \qquad \text{and} \qquad F_0(\xi) := S(\Phi)(\xi)$$

for $\Phi \in (\mathcal{S})^*$, $\xi \in \mathcal{S}$, paying attention to (i) of Proposition 2. From (6) we have $F_\theta(\xi) = F_0(e^{i\theta})\cdot \exp\left[ie^{i\theta}\sin\theta|\xi|^2\right]$. Since F_0 is Fréchet differentiable, the functional

$F_\theta(\xi)$ is differentiable in θ as well, and it is easy to check that

$$
\begin{aligned}
(17)\qquad & \lim_{\theta\to 0}\frac{1}{\theta}\{F_\theta(\xi)-F_0(\xi)\} \\
&= \langle F_0'(\mathrm{e}^{i\theta}\xi), i\mathrm{e}^{i\theta}\rangle \cdot \exp\Big(i\mathrm{e}^{i\theta}\sin\theta|\xi|^2\Big)\Big|_{\theta=0} \\
&\qquad + F_0(\mathrm{e}^{i\theta}\xi)\cdot\frac{d}{dt}\exp\Big(i\mathrm{e}^{i\theta}\sin\theta|\xi|^2\Big)\Big|_{\theta=0} \\
&= i\langle F'(\xi),\xi\rangle + i|\xi|^2\cdot F(\xi).
\end{aligned}
$$

While, we can easily check that the U-functional $\theta^{-1}\cdot\{F_\theta(\xi)-F_0(\xi)\}$, $\theta\in\mathbb{R}$ satisfies the uniform bounded criterion: $\exists C_0>0$ so that

$$
\sup_{\substack{z\in\mathbb{C}\\ |z|=R}}\left|\frac{1}{\theta}\{\tilde{F}_\theta(z\xi)-\tilde{F}_0(z\xi)\}\right| \le C_0\exp\big(c_1R^{c_2}|\xi|_p^2\big)
$$

holds for all $R>0$, all $\xi\in\mathcal{S}$ with $c_1>0, c_2>0$, where $\tilde{F}_*$ denotes an entire analytic extension of F. Hence, the strong convergence criterion theorem [25] (see also [11, Chapter 4]) allows convergence of

$$
S^{-1}\Big(\frac{1}{\theta}\{F_\theta(\cdot)-F_0(\cdot)\}\Big)(x)=\frac{1}{\theta}\{\Psi_\theta\Phi(x)-\Phi(x)\}
$$

in $(\mathcal{S})^*$ as θ tends to zero. We need the following two lemmas.

Lemma 13. *(cf. [11,Theorem 6.11,p.196]) Let $F(\xi)=S\Phi(\xi),\xi\in\mathcal{S}$ for $\Phi\in(\mathcal{S})^*$. Then*
(i) F is Fréchet differentiable;
(ii) the S-transform of $N\Phi(x)$ is given by $\langle F'(\xi),\xi\rangle$, $\xi\in\mathcal{S}$;
where N is the number operator.

Lemma 14. *(cf. [11,Theorem 6.20,p.206]) For any Φ in $(\mathcal{S})^*$, the S-transform of $\Delta_G^*\Phi(x)$ is given by $|\xi|^2\ S\Phi(\xi)$, $\xi\in\mathcal{S}$.*

We may deduce at once that

$$
(18)\qquad S(N\Phi+\Delta_G^*\Phi)(\xi)=\langle F'(\xi),\xi\rangle+|\xi|^2F(\xi),\quad \xi\in\mathcal{S},
$$

with simple applications of Lemma 13 and Lemma 14. Moreover, it is easily verified from (17) and (18) together with the above-mentioned convergence result that

$$
\begin{aligned}
\lim_{\theta\to 0}\frac{1}{\theta}(\Psi_\theta-Id)\Phi(x) &= \lim_{\theta\to 0}S^{-1}\Big(\frac{1}{\theta}\{F_\theta(\cdot)-F_0(\cdot)\}\Big)(x) \\
&= S^{-1}\big(i\langle F'(\xi),\xi\rangle+i|\xi|^2\cdot F(\xi)\big) \\
&= i(N+\Delta_G^*)\Phi(x), \qquad \text{in}\quad (\mathcal{S})^*,
\end{aligned}
$$

which completes the proof. □

§5. Application of PFM Transform

The purpose of this section is to show a typical example of application of the Pseudo-Fourier-Mehler transform Ψ_θ to the Cauchy problem.

Example 6. (A simple application of the PFM transform) Let us consider the following abstract Cauchy problem on the white noise space:

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= iNu(t,x) + \varphi(x), \\ u(0,\cdot) &= f(\cdot) \in (\mathcal{S}),\end{aligned} \tag{19}$$

with $t > 0$, where N denotes the number operator. One of the most remarkable benefits of white noise analysis consists in its application to differential equation theory and how to solve the problem (cf. [1], [2,3], [4,8]). Especially in [4,8], by resorting to the analogy in the finite dimensional cases we have applied the infinite dimensional Kuo type Fourier transform to the Cauchy problem for heat equation type with Gross Laplacian, and have succeeded in derivation of the general solution and also in direct verification for existence and uniqueness of the solution. On this account, we think of using the Fourier transform to the aforementioned problem. Recall the formula:

$$\mathcal{F}(N\Phi) = N(\mathcal{F}\Phi) + \Delta_G^*(\mathcal{F}\Phi), \qquad \text{for all } \Phi \in (\mathcal{S})^*. \tag{20}$$

We set $v(t,y) \equiv (\mathcal{F}u(t,\cdot))(y)$ for each $t \in \mathbb{R}_+$. We may employ the Fourier transform $\mathcal{F}$ for (19) so as to obtain

$$\begin{aligned}\frac{\partial v(t,y)}{\partial t} &= iNv(t,y) + i\Delta_G^* v(t,y) + \hat{\varphi}(y), \\ \text{with} \quad & v(0,y) = \hat{f}(y),\end{aligned} \tag{21}$$

because we made use of the formula (20) and set $\hat{F} = \mathcal{F}F$. The operator part of the Fourier transformed problem (21) is exactly equivalent to the infinitesimal generator of PFM transform with parameter t (see Theorem 12). Hence, the semigroup theory in functional equation theory allows immediately the following explicit exression of the solution in question:

$$v(t,y) = \Psi_t \hat{f}(y) + \int_0^t \Psi_{t-s}\hat{\varphi}(y)ds. \tag{22}$$

We can show the existence and uniqueness of the solution by applying Theorem 4 and Theorem 5 to (22) under a certain condition on the initial data φ, f. In that case the integral term appearing in (22) should be interpreted as Bochner type one. So much for the Cauchy problem, because this is not our main topic in this article. We shall go back to the PFM transform and proceed further in the next section.

§6. Intertwining Properties

In this section we shall investigate some intertwining properties between the Pseudo-Fourier-Mehler transform Ψ_θ and other typical operators in white noise analysis, such as Gâteaux differential, the adjoint of Gâteaux differential, Hida differential operator, and Kubo operator (the adjoint of Hida differential), etc. Furthermore, we shall introduce the characterization theorem for PFM transforms, which is one of our main results in this paper.

We begin with definition of the Gâteaux differential D_y in the direction $y \in \mathcal{S}^*$. For $y \in \mathcal{S}^*$ fixed, for the element φ in $(\mathcal{S})$ given by $\varphi(x) = \sum_{n=0}^\infty \langle : x^{\otimes n} :, f_n \rangle$, we put

$$D_y\varphi(x) = \lim_{\theta \to 0} \frac{\varphi(x+\theta y) - \varphi(x)}{\theta}, \qquad x \in \mathcal{S}^*. \tag{23}$$

The limit existence in the right hand side of (23) is always guaranteed, and $D_y\varphi(x)$ is actually given by

$$D_y\varphi(x) = \sum_{n=0}^\infty n \langle : x^{\otimes(n-1)} :, y \hat{\otimes}_1 f_n \rangle, \qquad x \in \mathcal{S}^*. \tag{24}$$

In fact, D_y becomes a continuous linear operator from $(\mathcal{S})$ into itself. Since the Dirac delta function δ_t lies in $\mathcal{S}^*$, adoption of δ_t instead of y does make sense in the above (23) and (24). On the other hand, the Hida differential operator ∂_t $(= \partial/\partial x(t))$ is originally proposed by T. Hida [9] and defined by

$$\partial_t := S^{-1} \frac{\delta}{\delta \xi(t)} S, \qquad \xi \in \mathcal{S}$$

(cf. [15]; see also [7]). It is well known that the action of ∂_t is equivalent to that of D_{δ_t} on the dense domain [11] (or [7],[14]). So we can define

$$\partial_t = D_{\delta_t}, \qquad t \in \mathbb{R}.$$

The Kubo operator ∂_t^* [15] is the adjoint of Hida differential ∂_t, defined by

$$\langle\langle \partial_t^* \Phi, \varphi \rangle\rangle = \langle\langle \Phi, \partial_t \varphi \rangle\rangle,$$

for $\Phi \in (\mathcal{S})^*$, $\varphi \in (\mathcal{S})$. As a matter of fact, ∂_t (resp. ∂_t^*) can be considered as a continuous linear operator from $(\mathcal{S})$ (resp. $(\mathcal{S})^*$) into itself with respect to the weak or strong topology. More precisely, the Hida differential proves to be a continuous mapping from $(\mathcal{S})_{p+q}$ into $(\mathcal{S})_q$ for $q > \frac{1}{4}$, $p \geq 0$, while the Kubo operator turns out to be the one from $(\mathcal{S})_{-p}$ into $(\mathcal{S})_{-(p+q)}$ for the same pair p, q as given above. For $\xi \in \mathcal{S}$, $\varphi \in (\mathcal{S})$, the derivative $(D_\xi\varphi)(x)$ is defined in the usual manner, and there exists its extension $\tilde{D}_\xi : (\mathcal{S})^* \to (\mathcal{S})^*$. Even for that, we shall henceforth use the same notation D_ξ for brevity, as far as there is no confusion in the context. We set $q_\xi := i(D_\xi + D_\xi^*)$, where D_ξ^* is the adjoint of D_ξ.

Lemma 15. *For each $\theta \in \mathbb{R}$, $t \in \mathbb{R}$,*

$$\Psi_\theta(\partial_t^* \Phi) = e^{i\theta} \partial_t^*(\Psi_\theta \Phi)$$

holds for all $\Phi \in (\mathcal{S})^*$.

Proof. First of all, note that $S(\partial_t^* \Phi)(\xi) = \xi(t) \cdot S(\Phi)(\xi)$. So, for the generalized white noise functional $\Phi \in (\mathcal{S})^*$ given in the form $\Phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle$, $x \in \mathcal{S}^*$ we readily get

$$S(\Psi_\theta(\partial_t^* \Phi))(\xi) = \mathrm{e}^{i\theta} \xi(t) \cdot \sum_{n=0}^{\infty} \langle f_n, \mathrm{e}^{in\theta} \xi^{\otimes n} \rangle \cdot \exp\bigl(i\mathrm{e}^{i\theta} \sin\theta |\xi|^2\bigr). \tag{25}$$

While we establish

$$S(\Psi_\theta(\partial_t^* \Phi))(\xi) = \mathrm{e}^{i\theta} S(\partial_t^*(\Psi_\theta \Phi))(\xi) \tag{26}$$

by applying (25), because we made use of the relation

$$S(\partial_t^*(\Psi_\theta \Phi))(\xi) = \xi(t) \cdot (S\Phi)(\mathrm{e}^{i\theta}\xi) \cdot \exp\bigl(i\mathrm{e}^{i\theta} \sin\theta |\xi|^2\bigr).$$

An application of the Potthoff-Streit characterization theorem (Theorem 1) to (26) leads to the required equality in Hida distribution sense. □

Proposition 16. *For each $\theta \in \mathbb{R}$, $t \in \mathbb{R}$*
(i) $\Psi_\theta(\partial_t \Phi) = e^{-i\theta} \partial_t(\Psi_\theta \Phi) - 2i \sin\theta \partial_t^*(\Psi_\theta \Phi)$;
(ii) $\Psi_\theta(x(t)\Phi) = e^{-i\theta} x(t)(\Psi_\theta \Phi)$;
hold for all $\Phi \in (\mathcal{S})^*$.

Remark 7. The assertion (i) of Proposition 16 follows from a direct computation. We have only to employ the following two rules:

$$S\partial_t(\cdot) = \frac{\delta}{\delta\xi(t)} S(\cdot), \qquad \partial_t^*(\cdot) = S^{-1}\xi(t)S(\cdot).$$

The second assertion (ii) is also due to a simple computation together with the first assertion (i) and Lemma 15. Moreover, we need to apply the multiplication operator: $x(t)(\cdot) = (\partial_t + \partial_t^*)\,(\cdot)$ (e.g. [19]). Those proofs go almost similarly as in the proof of Lemma 15 and are very easy, hence omitted.

The next proposition indicates some intertwining property between the PFM transform and Gâteaux differential operator.

Proposition 17. *For each parameter $\theta \in \mathbb{R}, t \in \mathbb{R}$*
(i) $e^{-i\theta} \tilde{D}_\xi(\Psi_\theta \Phi) = \Psi_\theta(\tilde{D}_\xi \Phi) + 2i \sin\theta \cdot D_\xi^*(\Psi_\theta \Phi)$;
(ii) $\tilde{D}_\xi(\Psi_\theta \Phi) + D_\xi^*(\Psi_\theta \Phi) = e^{i\theta} \Psi_\theta(\widetilde{\langle \cdot, \xi \rangle} \Phi)$;
hold for all generalized white noise functionals in $(\mathcal{S})^*$.

Proof. It is interesting to note that Gâteaux differential D_ξ and its adjoint D_ξ^* enjoy the integral kernel operator theoretical expressions in white noise analysis (see the next section; or [11,12], [23]). Namely,

$$\tilde{D}_\xi := \left(\int_{\mathbb{R}} \xi(t) \partial_t dt \right)^{\sim}, \quad \text{and} \quad D_\xi^* := \int_{\mathbb{R}} \xi(t) \partial_t^* dt, \quad \forall \xi \in \mathcal{S}. \tag{27}$$

Let $\Delta = \{t_k\}$ be a proper finite partition of the t parameter space, and $|\Delta|$ denotes the maximum of increment Δt_k over $1 \leq k \leq m$. The assertion (i) yields from (i) of Proposition 16. In fact, by linearity of the PFM transform we get

$$\sum_{k=1}^{m} \Delta t_k\, \xi(t_k) \cdot \Psi_\theta(\partial_{t_k}\Phi) = \Psi_\theta\Big(\sum_{k=1}^{m} \xi(t_k)\partial_{t_k}\Delta t_k \cdot \Phi\Big), \tag{28}$$

for $\forall \xi \in \mathcal{S}$. Consider the same type finite summation for the other terms in (i) of Proposition 16. By taking the limit $m \to \infty$ and by continuity of Ψ_θ (Theorem 10), we can obtain the desired result with consideration of Eq.(27).
As to (ii), note first that we can have the expression

$$\tilde{q}_\xi = i\widetilde{\langle x, \xi\rangle} = \left(i\int_{\mathbb{R}} x(t)\xi(t)dt\right)^{\sim}, \tag{29}$$

by virtue of the multiplication operator $x(t)(\cdot)$ (cf. Remark 7). With (ii) of Proposition 16, we may take advantage of continuity of Ψ_θ and (29) to deduce that

$$\begin{aligned}
\mathrm{e}^{-i\theta}(D_\xi + D_\xi^*)(\Psi_\theta\Phi) &= \mathrm{e}^{-i\theta}\Big(\int_{\mathbb{R}} x(t)\xi(t)dt\Big)(\Psi_\theta\Phi) \\
&= \lim_{m\to\infty} \Psi_\theta\Big(\sum_{k=0}^{m} \Delta t_k \xi(t_k)x(t_k)\cdot\Phi\Big) \\
&= \Psi_\theta(\langle x, \xi\rangle \cdot \Phi)
\end{aligned}$$

by passage to the limit $|\Delta| \to 0$. □

The following theorem gives the characterization for Pseudo-Fourier-Mehler transforms $\{\Psi_\theta;\ \theta \in \mathbb{R}\}$, which is one of our main results in this paper.

Theorem 18 [6]. *The Pseudo-Fourier-Mehler transform $\{\Psi_\theta; \theta \in \mathbb{R}\}$ satisfies the following conditions:*
(P1) $\Psi_\theta : (\mathcal{S})^* \to (\mathcal{S})^*$ is a continuous linear operator for $forall \theta \in \mathbb{R}$;
(P2) $\Psi_\theta(\tilde{D}_\xi\Phi) = e^{i\theta}\tilde{D}_\xi(\Psi_\theta\Phi) - 2\sin\theta \cdot \tilde{q}_\xi(\Psi_\theta\Phi)$;
(P3) $\Psi_\theta(\tilde{q}_\xi\Phi) = e^{-i\theta}\tilde{q}_\xi(\Psi_\theta\Phi)$;
where $\Phi \in (\mathcal{S})^*$, $\xi \in \mathcal{S}$. Conversely, if a continuous linear operator $A_\theta : (\mathcal{S})^* \to (\mathcal{S})^*$ satisfies the above conditions: (P1) $\sim$ (P3), then A_θ is a constant multiple of Ψ_θ.

Proof. (P1) is obvious (Theorem 10). (P2)(resp. (P3)) yields from (i)(resp. (ii)) of Proposition 17. It is due to a simple computation. Conversely, suppose that the operator A_θ be compatible with (P1),(P2) and (P3). We need the following results.

Lemma 19. *We assume that A_θ be a continuous linear operator from $(\mathcal{S})^*$ into itself, satisfying the three conditions (P1) $\sim$ (P3). Then the following relations*
(i) $(\Psi_\theta^{-1}\Xi_\theta)D_\xi = D_\xi(\Psi_\theta^{-1}\Xi_\theta)$;
(ii) $(\Psi_\theta^{-1}\Xi_\theta)q_\xi = q_\xi(\Psi_\theta^{-1}\Xi_\theta)$;
(iii) $(\Psi_\theta^{-1}\Xi_\theta)D_\xi^* = D_\xi^*(\Psi_\theta^{-1}\Xi_\theta)$;
hold for all $\xi \in \mathcal{S}$, $\theta \in \mathbb{R}$.

The proof will be given below. The next theorem is well known (e.g. [12, Theorem 3.6, p.267] or [23, Prop.5.7.6, p.148]).

Theorem 20. *Let Λ be a continuous linear operator on $(\mathcal{S})^*$, satisfying*
(i) $\Lambda\tilde{q}_\xi = \tilde{q}_\xi\Lambda$, for any $\xi \in \mathcal{S}$;
(ii) $\Lambda D_\xi^* = D_\xi^*\Lambda$, for any $\xi \in \mathcal{S}$.
Then the operator Λ is a scalar operator.

Thus, by taking (ii),(iii) of Lemma 19 into account, we may apply Theorem 20 for A_θ to obtain the assertion: $\Psi_\theta^{-1}A_\theta$ is a scalar operator. □

Proof of Lemma 19. Basically it is due to a direct computation. Each proof goes similarly, so we shall show only (iii) below. For the other two we will give just rough instructions. First of all, note that we have only to consider $\Psi_{-\theta}$ instead of Ψ_θ^{-1} by virtue of Proposition 3. As to (i), it is sufficient to calculate it with (P2) for both and (P3) for the PFM transform. As for (ii), simply (P3) for both A_θ and Ψ_θ. As to (iii), for $\forall\Phi \in (\mathcal{S})^*$, $\forall\xi \in \mathcal{S}$

$$(30) \qquad \begin{aligned}(\Psi_\theta^{-1}A_\theta)D_\xi^*\Phi &= -i\Psi_\theta^{-1}(A_\theta q_\xi)\Phi - \Psi_\theta^{-1}(A_\theta D_\xi)\Phi,\\ &= -\mathrm{e}^{i\theta}(\Psi_\theta^{-1}q_\xi)A_\theta\Phi - \mathrm{e}^{i\theta}(\Psi_\theta^{-1}D_\xi)A_\theta\Phi\end{aligned}$$

because we used a relation

$$(31) \qquad D_\xi^* = -i\tilde{q}_\xi - \tilde{D}_\xi$$

in the first equality and also employed (P2),(P3) in the second one. An application of (P2),(P3) to the last expression in (30), together with (31) again, gives

$$\begin{aligned}(30) &= -iq_\xi(\Psi_\theta^{-1}A_\theta)\Phi - D_\xi(\Psi_\theta^{-1}A_\theta)\Phi\\ &= (-iq_\xi - D_\xi)(\Psi_\theta^{-1}A_\theta)\Phi = D_\xi^*(\Psi_\theta^{-1}A_\theta)\Phi,\end{aligned}$$

which completes the proof. □

§7. Fock Expansion

Let $\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ denote the space of continuous linear operators from $(\mathcal{S})$ into $(\mathcal{S})^*$. The space $\hat{\mathcal{S}}'_{l,m}(\mathbb{R}^{l+m})$ is a symmtrized space of $\mathcal{S}'(\mathbb{R}^{l+m})$ with respect to the first l, and the second m variables independently. By virtue of the symbol characterization theorem for operators on white noise functionals [21](see also [23]), for the operator Π lying in $\mathcal{L}((\mathcal{S}),(\mathcal{S})^*)$ there exists uniquely a kernel distribution $\kappa_{l,m}$ in $\hat{\mathcal{S}}'_{l,m}(\mathbb{R}^{l+m})$ such that the operator Π may have the Fock expansion:

$$\Pi = \sum_{l,m=0}^{\infty} \Pi_{l,m}(\kappa_{l,m}).$$

Moreover, the series $\Pi\varphi$, $\varphi \in (\mathcal{S})$ converges in $(\mathcal{S})^*$ [21]. Generally, each component $\Pi_{l,m}$ of the Fock expansion has a formal integral expression:

$$\Pi_{l,m}(\kappa) = \int_{\mathbb{R}^{l+m}} \kappa(s_1,\cdots,s_l,t_1,\cdots,t_m)\cdot \\ \cdot\, \partial^*_{s_1}\cdots\partial^*_{s_l}\partial_{t_1}\cdots\partial_{t_m} ds_1\cdots ds_l dt_1\cdots dt_m.$$

Remark 8. We call it an integral kernel operator with kernel distribution κ. The theory of integral kernel operators and the general expansion theory in white noise analysis were proposed and have been developed enthusiastically by N. Obata [21-23] (see also [11]). Those topics are closely related to quantum stochastic calculus, which has been greatly investigated in chief by Hudson, Meyer, and Parthasarathy. More details on this topic will be found in, for instance, (i) K.R.Parthasarathy: *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, Basel, 1992; (ii) P.A.Meyer: *Quantum Probability for Probabilists*, Lecture Notes in Mathematics Vol.1538, Springer-Verlag, Heidelberg, 1993.

We shall give below two typical examples of the integral kernel operators in white noise analysis.

Example 9. (The number operator N) Let $\tau \in (\mathcal{S}\otimes\mathcal{S})^*$ be the trace operator defined by

$$\langle \tau, \xi\otimes\eta\rangle = \langle \xi,\eta\rangle, \qquad \xi,\eta\in\mathcal{S}.$$

The number operator N is usually expressed as

$$\int_{\mathbb{R}} \partial^*_t \partial_t dt$$

by Kuo's notation in white noise analysis. By the Obata theory, N has the following representation as a continuous linear operator from $(\mathcal{S})$ into itself, namely,

$$N = \Pi_{1,1}(\tau) = \int_{\mathbb{R}^2} \tau(s,t)\partial^*_s\partial_t ds dt.$$

Example 10. (The Gross Laplacian Δ_G) By the usual notation in white noise analysis we have the expression

$$\Delta_G = \int_{\mathbb{R}} \partial^2_t dt.$$

Then the Gross Laplacian Δ_G can be also expressed by

$$\Delta_G = \Pi_{0,2}(\tau) = \int_{\mathbb{R}^2} \tau(s_1, s_2)\partial_{s_1}\partial_{s_2} ds_1 ds_2$$

as a continuous linear operator from $(\mathcal{S})$ into $(\mathcal{S})$.

Let us consider the general expansion of our Pseudo-Fourier-Mehler transform. We may take advantage of Obata's integral kernel operator theory in order to obtain Fock expansion representations of Ψ_θ and its adjoint G_θ. That is to say,

Theorem 21. *For $\theta \in \mathbb{R}$, the PFM transform Ψ_θ and the adjoint operator G_θ have the following Fock expansions:*

$$(i)\quad \Psi_\theta = \sum_{l,m=0}^{\infty} \frac{1}{l!m!}(ie^{i\theta}\sin\theta)^l(e^{i\theta}-1)^m \cdot \Pi_{2l+m,m}(\tau^{\otimes l}\otimes\lambda_m);$$

$$(ii)\quad G_\theta = \sum_{l,m=0}^{\infty} \frac{1}{l!m!}(ie^{i\theta}\sin\theta)^m(e^{i\theta}-1)^l \cdot \Pi_{l,l+2m}(\lambda_l\otimes\tau^{\otimes m});$$

where the kernel $\lambda_m \in (\mathcal{S}^{\otimes 2m})^*$ is given by

$$\lambda_m := \sum_{i_1,i_2,\cdots,i_m=0}^{\infty} e_{i_1}\otimes\cdots\otimes e_{i_m}\otimes e_{i_1}\otimes\cdots\otimes e_{i_m}.$$

§8. Generalization

Let $GL((\mathcal{S}))$ be the group of all linear homeomorphisms from $(\mathcal{S})$ into $(\mathcal{S})$. Then we have

Proposition 22. *$\{G_\theta; \theta\in\mathbb{R}\}$ is a regular one parameter subgroup of $GL((\mathcal{S}))$ with infinitesimal generator $i(N+\Delta_G)$.*

Let us consider some generalization. Suggested by [1], for example we propose to define the generalized PFM transform X_θ, $\theta \in \mathbb{R}$ as operator on $(\mathcal{S})^*$ whose U-functional is given by

$$S(X_\theta\Phi)(\xi) = \langle\langle \Phi, \exp\Big(e^{\alpha\theta}\langle\cdot,\xi\rangle - \frac{1}{2}J(\alpha,\beta;\theta)|\xi|^2\Big)\rangle\rangle, \tag{32}$$

(cf. Eq. (7) in Definition 1 of PFM transform), for $\xi \in \mathcal{S}$, $\Phi \in (\mathcal{S})^*$. We set

$$\begin{aligned} &J(\alpha,\beta;\theta) = e^{2\alpha\theta} - 2H(\alpha,\beta;\theta),\\ &\text{with}\quad H(\alpha,\beta;\theta) = h(\alpha,\beta)\cdot(e^{2\alpha\theta}-1),\end{aligned}$$

where $h(\alpha,\beta) = \beta/2\alpha$, for α, $\beta\in\mathbb{C}$, $\alpha\neq 0$. Then we denote the adjoint operator of X_θ by Z_θ.

Claim 23. *The set $\{Z_\theta; \theta\in\mathbb{R}\}$ is a regular one parameter subgroup of $GL((\mathcal{S}))$.*

Claim 24. *The infinitesimal generator of* $\{Z_\theta; \theta \in \mathbb{R}\}$ *is given by the operator* $\alpha N + \beta \Delta_G$.

Claim 25. *The generalized PFM transform* $\{X_\theta;\ \theta \in \mathbb{R}\}$ *is a one parameter subgroup of* $GL((\mathcal{S})^*)$.

Claim 26. *The infinitesimal generator of* $\{X_\theta; \theta \in \mathbb{R}\}$ *is given by the operator* $\alpha N + \beta \Delta_G^*$.

Remark 11. The above definition (32) of generalized PFM transform X_θ can be alternatively replaced by the following expression:

$$S(X_\theta \Phi)(\xi) = F(\mathrm{e}^{\alpha\theta}\xi) \cdot \exp\Big(H(\alpha, \beta; \theta) \cdot |\xi|^2\Big),$$

where F denotes the U-functional of Φ in $(\mathcal{S})^*$, i.e., $S\Phi = F$.

Remark 12. Especially when $\alpha = \beta = i (\in \mathbb{C})$, then the above-defined generalized PFM transforms X_θ are, of course, attributed to the simple PFM transforms Ψ_θ given by (6), (7) in the section 3.

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